ISRAEL JOURNAL OF MATHEMATICS, Vol. 68, No. 2, 1989

INTERPOLATION OF COMPACTNESS USING ARONSZAJN–GAGLIARDO FUNCTORS

BY

FERNANDO COBOS[a] AND JAAK PEETRE[†,b]

[a]*Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain; and*

[b]*Matematiska Institutionen, Box 6701, S-11385 Stockholm, Sweden*

ABSTRACT

We prove that if $T: A_0 \to B_0$ and $T: A_1 \to B_1$ both are compact, then $T: F(\bar{A}) \to F(\bar{B})$ is also compact, where F is the minimal or the maximal functor in the sense of Aronszajn–Gagliardo. We also derive some results for ordered couples.

0. Introduction

Let us begin by briefly reviewing the history of our subject.

In 1960 Krasnosel'skiĭ [16] established a generalization of the Riesz–Thorin theorem, to the effect that if T is a linear operator such that $T: L_{p_0} \to L_{q_0}$ and $T: L_{p_1} \to L_{q_1}$ the latter map being not just bounded but also compact, then $T: L_p \to L_q$ is compact too, this in the usual hypotheses about the exponents but also assuming that $q_0 < \infty$.

Krasnosel'skiĭ's theorem leads to the question whether similar results hold true in abstract interpolation, replacing (L_{p_0}, L_{p_1}) and (L_{q_0}, L_{q_1}) by general (compatible) Banach pairs (A_0, A_1) and (B_0, B_1). The complete answer to this question is not yet known.

The first partial results were published in 1964 by Lions and one of the present authors [18] (see also [11]), for the case $A_0 = A_1$ or $B_0 = B_1$, and by A. Persson [23], for the general case $A_0 \neq A_1$ and $B_0 \neq B_1$; the latter had however to suppose that the couple (B_0, B_1) satisfies a certain approximation assumption, corresponding to Krasnosel'skiĭ's assumption $q_0 < \infty$. An approx-

[†] Supported in part by Ministerio de Educación y Ciencia (SAB-87-0172).

Received November 7, 1988 and in revised form June 1, 1989

imation condition of the same type was also used in 1966 by Kreĭn and Petunin [17] to derive a compactness theorem between scales of Banach spaces.

In 1969 Hayakawa [13] found a result for the real method without any approximation hypothesis. However, he had to assume that both $T: A_0 \to B_0$ and $T: A_1 \to B_1$ are compact. Quite recently a somewhat novel approach to Hayakawa type ("twosided") theorems has been developed in [6] and [7], the former paper dealing with the K-method and the latter with the J-method. In addition, these papers give Krasnosel'skiĭ type ("onesided") results without approximation hypothesis but for "ordered"† pairs (meaning that one space is contained in the other).

In this paper we prove similar results to the ones in [6] and [7], in particular thus twosided interpolation theorems without approximation hypothesis, for the minimal ("orbit") and the maximal ("coorbit") functors in the sense of Aronszajn–Gagliardo [1]. When specialized this gives back not only the results of these two papers but also Krasnosel'skiĭ type results for those cases of ordered pairs which were not considered in [6] or in [7]. We also derive corresponding twosided and onesided theorems for other "concrete" interpolation methods such as the " $\pm$ " method (see [22], [12]). Such results might potentially be interesting, e.g. applications to integral operators in Orlicz spaces (see again [12]). However, contrary to what we thought at an early stage of this investigation, our techniques do not seem to apply in the case of the complex method (see [5]).

We start by establishing compactness results for a general functor when one of the Banach couples satisfies a certain approximation condition. This is done in Section 1. Unlike what happens with Persson's condition [23], which is only useful when applied to the second couple, we use an approximation hypothesis that works in both positions (the first or the second pair). In Sections 2 and 3 we then specialize these general results to the case of Aronszajn–Gagliardo functors. In this way we derive compact interpolation theorems for the orbit and coorbit functors *without* any approximation condition on either of the couples (A_0, A_1) and (B_0, B_1).

The secret of our approach is that when working systematically with Aronszajn–Gagliardo functors the approximation hypothesis is so to speak shifted from the given pair to the one of which it is the orbit or coorbit. In this

† The term "ordered" in this sense is used in the forthcoming monograph [4].

respect we have been guided by a similar treatment of Wolff's theorem [25] in [15], [8]. Once again one thus has an illustration of the old truth that in mathematics things sometimes become simpler if one passes to a more general setting ... .

Finally, in Section 4, we discuss some open questions, in particular whether it is possible to prove general onesided results (of the Krasnosel'skiĭ type) without approximation hypothesis.

The authors would like to thank M. Cwikel for his helpful comments on the first version of this article.

1. General twosided interpolation theorems for compact operators

Let us start by fixing the terminology (see [3]).

Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two Banach couples. We write $T: \bar{A} \to \bar{B}$ to mean that T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restriction to each A_i defines a bounded operator from A_i into B_i $(i = 0, 1)$. We put

$$\| T \|_{\bar{A},\bar{B}} = \max\{ \| T \|_{A_0,B_0}, \| T \|_{A_1,B_1} \}.$$

For convenience we shall work with interpolation functors F of exponent θ, for some $0 < \theta < 1$. Thus F associates to each Banach couple $\bar{A}$ an intermediate Banach space $F(\bar{A})$, i.e. $A_0 \cap A_1 \hookrightarrow F(\bar{A}) \hookrightarrow A_0 + A_1$, in such a way that given any other Banach couple $\bar{B}$ and any operator $T: \bar{A} \to \bar{B}$, the restriction of T to $F(\bar{A})$ defines a bounded operator from $F(\bar{A})$ into $F(\bar{B})$ and the following holds:

$$\| T \|_{F(\bar{A}),F(\bar{B})} \leqq C \| T \|_{A_0,B_0}^{1-\theta} \| T \|_{A_1,B_1}^{\theta} \tag{1}$$

for some constant C independent of T.

In order to describe an extremal property of the real interpolation method $\bar{A}_{\theta,q}$ (see [18]) with respect to this class of functors, we recall that the K-functional is defined by

$$K(t, a; \bar{A}) = \inf\{ \| a_0 \|_{A_0} + t \| a_1 \|_{A_1} : a = a_0 + a_1, a_i \in A_i \}, \ a \in A_0 + A_1, \ t > 0.$$

The next lemma shows that

$$\bar{A}_{\theta,1} \hookrightarrow F(\bar{A}) \hookrightarrow \bar{A}_{\theta,\infty}.$$

LEMMA 1.1. *Let F be an interpolation functor of exponent θ $(0 < \theta < 1)$ and let $\bar{A} = (A_0, A_1)$ be any Banach couple. Then we have*

$$\| a \|_{F(\bar{A})} \leqq C \| a \|_{A_0}^{1-\theta} \| a \|_{A_1}^{\theta}, \qquad a \in A_0 \cap A_1, \tag{2}$$

(3) $$K(t, a; \bar{A}) \leqq Ct^{\theta} \| a \|_{F(\bar{A})}, \qquad a \in F(\bar{A}), \quad t > 0,$$

for some constant C independent of a and t.

PROOF. This result is known, see [3], Thm. 3.9.1. Since inequality (3) is established there with the additional assumption

"$A_0 \cap A_1$ dense in A_0 and in A_1"

we show now how to derive it without that supposition.

Let a be any element in $F(\bar{A})$ and let t be any positive number. Applying the Hahn–Banach theorem we can find f belonging to the dual space of $A_0 + A_1$ such that

$$\langle f, a \rangle = K(t, a)$$

and

$$\| f \|_{A_0'} \leqq 1, \qquad \| f \|_{A_1'} \leqq t.$$

It follows from (1) that $f: F(\bar{A}) \to \mathbf{C}$ with norm less than a constant multiplied by t^{θ}. Therefore we conclude that

$$K(t, a) \leqq Ct^{\theta} \| a \|_{F(\bar{A})}. \qquad \square$$

As a consequence of Lemma 1.1, it follows that the result of Lions–Peetre ([18] or [3], 3.8) mentioned in the Introduction works for the spaces produced by the functor F. We state it now for later use.

LIONS–PEETRE LEMMA. *Let F be interpolation functor of exponent θ $(0 < \theta < 1)$, let $\bar{A} = (A_0, A_1)$ be a Banach couple and let B be a Banach space. Assume that T is a linear operator.*

(i) *If $T: A_0 \to B$ is bounded and $T: A_1 \to B$ is compact, then $T: F(\bar{A}) \to B$ is compact.*

(ii) *If $T: B \to A_0$ is bounded and $T: B \to A_1$ is compact, then $T: B \to F(\bar{A})$ is compact.*

To proceed to our compactness theorems, we first assume that one of the Banach couples satisfies a certain approximation hypothesis. Afterwards we shall see that this condition is not needed in many applications.

Consider the following *approximation condition* on the Banach couple $\bar{A} = (A_0, A_1)$:

There exists a sequence $\{P_n\}_{n=1}^{\infty}$ of linear maps from $A_0 + A_1$ into $A_0 \cap A_1$ and two other sequences $\{Q_n^+\}_{n=1}^{\infty}$, $\{Q_n^-\}_{n=1}^{\infty}$ of linear maps from $A_0 + A_1$ into $A_0 + A_1$ such that:

(I) They are uniformly bounded in $\bar{A}$,

$$\sup_n \{ \| P_n \|_{\bar{A},\bar{A}}, \| Q_n^+ \|_{\bar{A},\bar{A}}, \| Q_n^- \|_{\bar{A},\bar{A}} \} < \infty.$$

(II) The identity operator I on $\bar{A}$ can be decomposed as

$$I = P_n + Q_n^+ + Q_n^-, \qquad n = 1, 2, \ldots .$$

(III) For each positive integer n one has

$$Q_n^+ : A_0 \to A_1 \quad \text{and} \quad Q_n^- : A_1 \to A_0$$

and the sequences of norms

$$\{ \| Q_n^+ \|_{A_0,A_1} \}_{n=1}^{\infty}, \qquad \{ \| Q_n^- \|_{A_1,A_0} \}_{n=1}^{\infty}$$

converge to 0 when $n \to \infty$.

Remark 1.2. Note the difference between this approximation condition and the one used by A. Persson in [23]. He only considers a sequence of linear maps $\{R_n\}_{n=1}^{\infty}$ from $A_0 + A_1$ into $A_0 \cap A_1$, uniformly bounded in $\bar{A}$ and such that

$$\| a - R_n a \|_{A_0} \to 0 \quad \text{as } n \to \infty, \quad \text{for each } a \in A_0.$$

In our case, if $A_0 \cap A_1$ is dense in $F(\bar{A})$, we also have a convergence statement of this type but for $F(\bar{A})$:

$$\| a - P_n a \|_{F(\bar{A})} \to 0 \quad \text{as } n \to \infty, \quad \text{for each } a \in F(\bar{A}).$$

Indeed, let $a \in A_0 \cap A_1$. Then we get

$$\begin{aligned} \| Q_n^+ a \|_{F(\bar{A})} &\leqq \| Q_n^+ \|_{A_0,F(\bar{A})} \| a \|_{A_0} \\ &\leqq C \| Q_n^+ \|_{A_0,A_0}^{1-\theta} \| Q_n^+ \|_{A_0,A_1}^{\theta} \| a \|_{A_0}. \end{aligned}$$

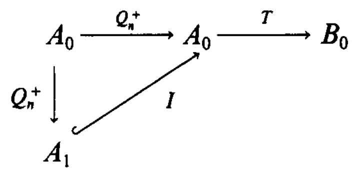
It follows from (I) and (III) that $\| Q_n^+ a \|_{F(\bar{A})} \to 0$. In the same way we prove that $\| Q_n^- a \|_{F(\bar{A})} \to 0$, and therefore $\| a - P_n a \|_{F(\bar{A})} \to 0$. Making appeal to the density concludes the proof.

Some consequences of this approximation condition are given in the next lemmas.

Lemma 1.3. *Let the hypotheses* (I) *to* (III) *be fulfilled for* $\bar{A} = (A_0, A_1)$, *let* $\bar{B} = (B_0, B_1)$ *be any other Banach couple and let* $T : \bar{A} \to \bar{B}$. *If* A_1 *is continuously embedded in* A_0 *or alternatively* B_1 *continuously embedded in* B_0, *then*

$$\| TQ_n^+ \|_{A_0,B_0} \to 0 \qquad as\ n \to \infty.$$

PROOF. Assume first $A_1 \hookrightarrow A_0$. We can factorize TQ_n^+ by means of the following diagram:



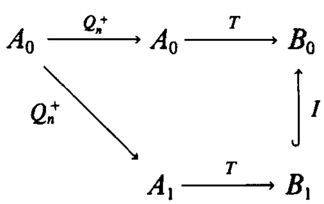
Thus we have

$$\| TQ_n^+ \|_{A_0,B_0} \leqq \| T \|_{A_0,B_0} \| I \|_{A_1,A_0} \| Q_n^+ \|_{A_0,A_1}.$$

Whence hypothesis (III) implies that

$$\| TQ_n^+ \|_{A_0,B_0} \to 0 \qquad \text{as } n \to \infty.$$

In the case $B_1 \hookrightarrow B_0$ the proof is similar, using now the factorization



□

Given any Banach couple $\bar{A} = (A_0, A_1)$, we denote the closure of $A_0 \cap A_1$ in A_i $(i = 0, 1)$ by A_i° and we write $\bar{A}^\circ = (A_0^\circ, A_1^\circ)$. We use these closures in the following result:

LEMMA 1.4. *Assume that* $\bar{A} = (A_0, A_1)$ *satisfies the approximation hypotheses* (I) *to* (III). *The following holds*:

(i) *If* $a \in A_0^\circ$, *then* $\| Q_n^- a \|_{A_0} \to 0$ *as* $n \to \infty$.

(ii) *If* $a \in A_1^\circ$, *then* $\| Q_n^+ a \|_{A_1} \to 0$ *as* $n \to \infty$.

PROOF. First let $a \in A_0 \cap A_1$. We have

$$\| Q_n^- a \|_{A_0} \leqq \| Q_n^- \|_{A_1,A_0} \| a \|_{A_1}.$$

Hence hypothesis (III) yields

$$\| Q_n^- a \|_{A_0} \to 0 \qquad \text{as } n \to \infty.$$

The conclusion follows by taking into account that the maps $\{Q_n^-\}$ are uniformly bounded in A_0 and that $A_0 \cap A_1$ is dense in A_0°.

The proof of (ii) is symmetrical. □

Our last auxiliary result has a proof analogous to Lemma 1.3. We omit the details.

LEMMA 1.5. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let the hypotheses* (I) *to* (III) *be fulfilled by $\bar{B} = (B_0, B_1)$ and let $T : \bar{A} \to \bar{B}$. If either A_0 is continuously embedded in A_1 or B_0 is continuously embedded in B_1, then*

$$\| Q_n^- T \|_{A_0, B_0} \to 0 \qquad \textit{as } n \to \infty.$$

We are now able to establish the crucial results of this section.

THEOREM 1.6. *Assume that $\bar{A} = (A_0, A_1)$ satisfies the approximation hypotheses* (I) *to* (III) *and let $\bar{B} = (B_0, B_1)$ be any other Banach couple. If $T : \bar{A} \to \bar{B}$ with*

$$T : A_i \to B_i \quad \textit{compactly for } i = 0, 1,$$

then

$$T : F(\bar{A}) \to F(\bar{B}) \quad \textit{is also compact},$$

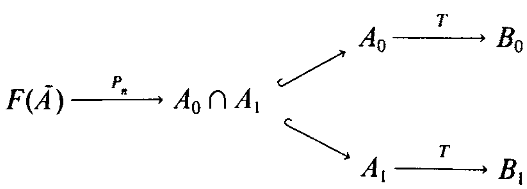
where F is any interpolation functor of exponent θ $(0 < \theta < 1)$.

PROOF. To see that $T : F(\bar{A}) \to F(\bar{B})$ is compact it suffices to show two things:

(1) $TP_n : F(\bar{A}) \to F(\bar{B})$ is compact for each $n = 1, 2, \ldots$.

(2) $\| T - TP_n \|_{F(\bar{A}), F(\bar{B})} \to 0$ as $n \to \infty$.

The proof of (1) is immediate as we can factorize TP_n using the following diagram:



Hence, since $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are both compact, the compactness of TP_n follows by Lions–Peetre Lemma, clause (ii). (In fact, in the situation at hand, the compactness of TP_n can be checked directly in a really simple way, as we have compactness on the two sides.)

To prove (2) we write using (II)

$$T - TP_n = T(I - P_n) = TQ_n^+ + TQ_n^- .$$

It is sufficient to show that, say,

$$\| TQ_n^+ \|_{F(\bar{A}),F(\bar{B})} \to 0,$$

as the corresponding statement with the second term (Q_n^+ replaced by Q_n^-) then follows by symmetry. In view of our assumption on the functor F, we have

$$\| TQ_n^+ \|_{F(\bar{A}),F(\bar{B})} \leqq C \| TQ_n^+ \|_{A_0,B_0}^{1-\theta} \| TQ_n^+ \|_{A_1,B_1}^{\theta}.$$

In this inequality, both factors on the right are bounded, so it suffices to check that the former tends to 0. Assume the contrary. Then we can find a subsequence $\{TQ_{n_1}^+\}$ of $\{TQ_n^+\}$ and a bounded sequence $\{a_{n_1}\} \subset A_0$ such that

$$\| TQ_{n_1}^+ a_{n_1} \|_{B_0} \to \lambda \neq 0.$$

By the compactness of $T: A_0 \to B_0$ and uniform boundedness of $\{Q_n^+\}$ on A_0, we may also assume, passing to another subsequence if necessary, that $\{TQ_{n_2}^+ a_{n_2}\}$ converges to some element b in B_0, so that $\| b \|_{B_0} = \lambda$. But by hypothesis (III) we obtain that $\{TQ_{n_2}^+ a_{n_2}\}$ converges to 0 in B_1. Therefore $b = 0$ contradicting $\lambda \neq 0$. □

If either $\bar{A}$ or $\bar{B}$ is an ordered couple, then we do not need compactness in $T: A_0 \to B_0$ to derive that the interpolated operator is still compact:

THEOREM 1.7. *We make the same assumptions on $\bar{A}$, $\bar{B}$ and F as in the previous theorem. We assume also that A_1 is continuously embedded in A_0 or alternatively B_1 is continuously embedded in B_0, and that $T: \bar{A} \to \bar{B}$ with*

$$T: A_1 \to B_1 \quad \textit{compactly}.$$

Then

$$T: F(\bar{A}) \to F(\bar{B}) \quad \textit{is compact}.$$

PROOF. The compactness of $T: A_1 \to B_1$ and similar arguments to those in the proof of the previous theorem imply that

$$TP_n: F(\bar{A}) \to F(\bar{B}) \quad \text{is compact for each } n = 1, 2, \dots$$

and that

$$\| TQ_n^- \|_{A_1,B_1} \to 0 \quad \text{as } n \to \infty.$$

On the other hand, Lemma 1.3 gives that

$$\| TQ_n^+ \|_{A_0,B_0} \to 0 \quad \text{as } n \to \infty.$$

Since

$$\begin{aligned}\| T - TP_n \|_{F(\bar A),F(\bar B)} &\leqq \| TQ_n^+ \|_{F(\bar A),F(\bar B)} + \| TQ_n^- \|_{F(\bar A),F(\bar B)}\\ &\leqq C(\| TQ_n^+ \|_{A_0,B_0}^{1-\theta} \| TQ_n^+ \|_{A_1,B_1}^{\theta} + \| TQ_n^- \|_{A_0,B_0}^{1-\theta} \| TQ_n^- \|_{A_1,B_1}^{\theta})\end{aligned}$$

and the sequences $\{ \| TQ_n^+ \|_{A_1,B_1}\}$, $\{ \| TQ_n^- \|_{A_0,B_0}\}$ are bounded, we conclude that T is the norm limit of the sequence of compact operators $\{TP_n\}$. Therefore $T: F(\bar A) \to F(\bar B)$ is compact. □

Next we discuss the behaviour of compactness under interpolation when the second couple satisfies the approximation condition.

THEOREM 1.8. *Let $\bar A = (A_0, A_1)$ be a Banach couple and assume that $\bar B = (B_0, B_1)$ satisfies the approximation hypotheses* (I) *to* (III). *If $T: \bar A \to \bar B$ with*

$$T: A_i \to B_i \quad \textit{compactly for } i = 0, 1,$$

then

$$T: F(\bar A^\circ) \to F(\bar B^\circ) \quad \textit{is also compact},$$

where F is any interpolation functor of exponent θ $(0 < \theta < 1)$.

PROOF. Recall that A_i° is the closure of $A_0 \cap A_1$ in A_i. Hence

$$T: A_i^\circ \to B_i^\circ \qquad (i = 0, 1)$$

is still compact because $T(A_i^\circ) \subset B_i^\circ$.

Again, in order to show that $T: F(\bar A^\circ) \to F(\bar B^\circ)$ compactly, it suffices to establish that

(a) $P_nT: F(\bar A^\circ) \to F(\bar B^\circ)$ is compact for each $n = 1, 2, \ldots$ and

(b) $\| T - P_nT \|_{F(\bar A^0),F(\bar B^0)} \to 0$ as $n \to \infty$.

To prove (a) we have, this time, to use the Lions–Peetre Lemma, clause (i). In fact, the following diagram holds:

$$\begin{array}{ccccc} A_0^\circ & \xrightarrow{\;T\;} & B_0^\circ & & \\ & & & \searrow^{P_n} & \\ & & & & B_0 \cap B_1 \hookrightarrow F(\bar B^0) \\ & & & \nearrow_{P_n} & \\ A_1^\circ & \xrightarrow{\;T\;} & B_1^\circ & & \end{array}$$

where $T: A_i^\circ \to B_i^\circ$ is compact $(i = 0, 1)$. Thus by the said lemma, P_nT is also compact as a map from $F(\bar A^\circ)$ into $F(\bar B^\circ)$.

For the proof of (b) first we write by (II)

$$T - P_n T = (I - P_n)T = Q_n^- T + Q_n^+ T$$

and we may treat each term here separately. To see that

$$\| Q_n^- T \|_{F(\bar{A}^\circ), F(\bar{B}^\circ)} \to 0 \qquad \text{as } n \to \infty$$

we note that

$$\| Q_n^- T \|_{F(\bar{A}^\circ), F(\bar{B}^\circ)} \leqq C \| Q_n^- T \|_{A_0^\circ, B_0^\circ}^{1-\theta} \| Q_n^- T \|_{A_1^\circ, B_1^\circ}^{\theta}$$

both factors being bounded to the right in this inequality. Thus we only need to show that the former tends to 0. With this aim, let $\varepsilon > 0$ be given arbitrarily. In virtue of $T: A_0^\circ \to B_0^\circ$ compactly, we can find a finite subset $\{a_1, \ldots, a_m\}$ of the closed unit ball of A_0° such that for any $a \in A_0^\circ$ with $\| a \|_{A_0^\circ} \leqq 1$, we have

$$\min_{1 \leqq j \leqq m} \| Ta - Ta_j \|_{B_0^\circ} \leqq \varepsilon / 2C.$$

Here C is the constant of the hypothesis (I). On the other hand, by Lemma 1.4, clause (i), there is $N \in \mathbf{N}$ such that if $n \geqq N$ then

$$\| Q_n^- Ta_j \|_{B_0^\circ} \leqq \varepsilon / 2 \qquad (1 \leqq j \leqq m).$$

Consequently, given any $a \in A_0^\circ$ with $\| a \|_{A_0^\circ} \leqq 1$, we obtain for $n \geqq N$

$$\begin{aligned} \| Q_n^- Ta \|_{B_0^\circ} &\leqq \| Q_n^- (Ta - Ta_j) \|_{B_0^\circ} + \| Q_n^- Ta_j \|_{B_0^\circ} \\ &\leqq C\varepsilon/2C + \varepsilon/2 = \varepsilon, \end{aligned}$$

i.e.

$$\| Q_n^- T \|_{A_0^\circ, B_0^\circ} \leqq \varepsilon \qquad \text{if } n \geqq N.$$

With a similar reasoning, but now using the fact that $T: A_1^\circ \to B_1^\circ$ compactly, and Lemma 1.4, clause (ii), one can see that

$$\| Q_n^+ T \|_{F(\bar{A}^\circ), F(\bar{B}^\circ)} \to 0 \qquad \text{as } n \to \infty.$$

This gives (b) and completes the proof. □

In the case of ordered couples, we obtain again a onesided result.

Theorem 1.9. *We make the same assumptions on $\bar{A}$, $\bar{B}$ and F as in Theorem 1.8. We suppose further that either A_0 is continuously embedded in A_1 or B_0 is continuously embedded in B_1, and that $T: \bar{A} \to \bar{B}$ with*

$$T: A_1 \to B_1 \quad \textit{compactly}.$$

Then

$$T: F(\bar{A}^0) \to F(\bar{B}^0) \quad \textit{is compact}.$$

PROOF. Proceeding similarly to Theorem 1.8, we can check that

$$P_n T: F(\bar{A}^0) \to F(\bar{B}^0) \quad \text{is compact for each } n = 1, 2, \ldots$$

and that

$$\| Q_n^+ T \|_{F(\bar{A}^0), F(\bar{B}^0)} \to 0 \qquad \text{as } n \to \infty,$$

because only the compactness of $T: A_1^\circ \to B_1^\circ$ is needed in the arguments. In addition, by Lemma 1.5, we have

$$\| Q_n^- T \|_{A_0^\circ, B_0^\circ} \to 0 \qquad \text{as } n \to \infty.$$

This together with the boundedness of $\{ \| Q_n^- T \|_{A_1^\circ, B_1^\circ} \}$ imply that

$$\| Q_n^- T \|_{F(\bar{A}^0), F(\bar{B}^0)} \to 0 \qquad \text{as } n \to \infty.$$

Whence, using the estimate

$$\| T - P_n T \|_{F(\bar{A}^0), F(\bar{B}^0)} \leqq \| Q_n^- T \|_{F(\bar{A}^0), F(\bar{B}^0)} + \| Q_n^+ T \|_{F(\bar{A}^0), F(\bar{B}^0)},$$

we derive that $T: F(\bar{A}^o) \to F(\bar{B}^o)$ is compact. □

REMARK 1.10. The main examples of interpolation functors of exponent θ are the real method [18] and the complex one [5]. These two functors satisfy

$$F(A_0, A_1) = F(A_0^\circ, A_1^\circ) \tag{4}$$

for any Banach couple (A_0, A_1).

REMARK 1.11. In a more general way, any Aronszajn–Gagliardo minimal functor [1]

$$G[X_0, X_1; X](-)$$

fulfils (4) whenever $X_0 = X_0^\circ$ and $X_1 = X_1^\circ$ (see [14], Lemma 2/(i)). We shall deal with minimal functors in the next section.

2. The case of the Aronszajn–Gagliardo minimal functor

In what follows, we designate by $\bar{X} = (X_0, X_1)$ a fixed Banach couple and by X a fixed intermediate space for $\bar{X}$, i.e. $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$.

Let us recall the definition of the Aronszajn–Gagliardo minimal ("orbit") functor [1] (cf. [14], [19]) in a form suitable for our purposes.

Given $\bar{A} = (A_0, A_1)$ any Banach couple, let

$$U = U(\bar{A}) = \{S \mid S : \bar{X} \to \bar{A},\ \| S \|_{\bar{X},\bar{A}} \leqq 1\}$$

be the unit ball in the Banach space of all linear maps from the Banach pair $\bar{X}$ into the Banach pair $\bar{A}$.

If Y is any one of the spaces X, X_0, X_1, $X_0 \cap X_1$, $X_0 + X_1$ we denote by $l^1[Y] = l^1(U(\bar{A}), Y)$ the Banach space of all absolutely summable families $x = \{x_S\}$ of elements of Y indexed by the elements S of U:

$$x = \{x_S\} \in l^1[Y] \qquad \text{if and only if } \sum_{S \in U} \| x_S \|_Y < \infty.$$

It is convenient to set

$$\pi x = \sum_{S \in U} S x_S.$$

Clearly $x \in l^1[X_0]$ implies $\pi x \in A_0$ and if $x \in l^1[X_1]$ then $\pi x \in A_1$.

We now define

$$G(\bar{A}) = G[\bar{X}; X](\bar{A}) = \{a \mid a \in A_0 + A_1,\ \exists\, x \in l^1[X],\ a = \pi x\};$$

it is a Banach space in the natural quotient norm (as a quotient of $l^1[X]$) and if $T : \bar{A} \to \bar{B}$ then the restriction of T to $G(\bar{A})$ defines a bounded operator from $G(\bar{A})$ to $G(\bar{B})$. Thus G is an interpolation functor.

For convenience we restrict our attention to the case when G is an interpolation functor of exponent θ (where $0 < \theta < 1$), i.e.

$$\| T \|_{G(\bar{A}),G(\bar{B})} \leqq C \| T \|_{A_0,B_0}^{1-\theta} \| T \|_{A_1,B_1}^{\theta}.$$

In order to specialize the general results in Section 1 to the orbit functor, we shall need the following fact.

Lemma 2.1. *Let, in this context, $l^1(\cdot)$ denote l^1 with respect to any given index set $\mathcal{N}$ (i.e. if A is any Banach space then $l^1(A)$ consists of all A-valued families $a = \{a_\nu\}$ with $\sum_{\nu \in \mathcal{N}} \| a_\nu \|_A < \infty$). Then for any Banach couple $\bar{A} = (A_0, A_1)$*

$$l^1(G(\bar{A})) \hookrightarrow G(l^1(A_0), l^1(A_1))$$

holds.

Proof. Let $a = \{a_\nu\} \in l^1(G(\bar{A}))$. Then we may write

$$a_\nu = \sum_{S \in U} S x_{S,\nu} \qquad \text{with } \sum_{\substack{S \in U \\ \nu \in \mathcal{N}}} \| x_{S,\nu} \|_X < \infty.$$

Define linear maps S_ν which assign to an element $x \in X_0 + X_1$ a family which has the entry Sx at position ν and 0 elsewhere. It is clear that

$$S_\nu : X_0 \to l^1(A_0) \quad \text{and} \quad S_\nu : X_1 \to l^1(A_1)$$

with norm at most 1. As the above formula may be rewritten tautologically as

$$a = \sum_{\substack{S \in U \\ \nu \in \mathcal{N}}} S_\nu x_{S,\nu}$$

it follows that $a \in G(l^1(A_0), l^1(A_1))$, with the embedding from $l^1(G(\bar{A}))$ into $G(l^1(A_0), l^1(A_1))$ being continuous. □

We *impose* now the approximation hypotheses (I) to (III) on the Banach couple $\bar{X} = (X_0, X_1)$ that we have taken to define the functor G. Then the maps P_n *extend in a natural way* to maps from $l^1[X_0 + X_1]$ into $l^1[X_0 \cap X_1]$, the maps $\{Q_n^+\}$, $\{Q_n^-\}$ also extend to maps into $l^1[X_0 + X_1]$, *and the new maps preserve properties* (I) *to* (III). Indeed, let us check, for example, that

$$\| Q_n^+ \|_{l^1[X_0], l^1[X_1]} \to 0 \qquad \text{as } n \to \infty. \tag{1}$$

Given any $x = \{x_S\} \in l^1[X_0]$ we have

$$\begin{aligned} \| Q_n^+ x \|_{l^1[X_1]} &= \sum_{S \in U} \| Q_n^+ x_S \|_{X_1} \\ &\leq \| Q_n^+ \|_{X_0, X_1} \sum_{S \in U} \| x_S \|_{X_0} \\ &= \| Q_n^+ \|_{X_0, X_1} \| x \|_{l^1[X_0]}. \end{aligned}$$

This gives (1).

Now we are ready to state the compactness results for the functor G.

Theorem 2.2. *Let the hypotheses* (I) *to* (III) *be fulfilled for $\bar{X}$ and assume also that $G[\bar{X}; X](-)$ is an interpolation functor of exponent θ $(0 < \theta < 1)$. Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be any two Banach couples, and let $T : \bar{A} \to \bar{B}$ be such that*

$$T : A_0 \to B_0 \quad \textit{and} \quad T : A_1 \to B_1$$

are both compact. Then

$$T : G(\bar{A}) \to G(\bar{B}) \quad \textit{is compact}.$$

Proof. It is clear that

$$T : G(\bar{A}) \to G(\bar{B}) \quad \text{is compact}$$

if and only if

$$\tilde{T} = T\pi : l^1[X] \to G(\bar{B}) \quad \text{is compact.}$$

The relevant diagram to have in mind is

$$\begin{array}{c} l^1[X_0] \xrightarrow{\pi} A_0 \xrightarrow{T} B_0 \\ l^1[X_1] \xrightarrow{\pi} A_1 \xrightarrow{T} B_1 \\ \hline\hline l^1[X] \xrightarrow{\pi} G(\bar{A}) \xrightarrow{T} G(\bar{B}) \end{array}$$

Since $(l^1[X_0], l^1[X_1])$ satisfies the approximation condition, applying Theorem 1.6 we have that

$$\tilde{T} : G(l^1[X_0], l^1[X_1]) \to G(\bar{B})$$

is compact. In addition, according to Lemma 2.1,

$$l^1[X] \hookrightarrow l^1[G(X_0, X_1)] \hookrightarrow G(l^1[X_0], l^1[X_1]).$$

Therefore we conclude that

$$\tilde{T} : l^1[X] \to G(\bar{B}) \quad \text{is compact.} \qquad \square$$

Note that in the previous theorem neither $\bar{A}$ nor $\bar{B}$ is required to satisfy any approximation condition.

Using Theorem 1.7 instead of Theorem 1.6, we obtain the following onesided result.

THEOREM 2.3. *Let $\bar{X}$, G, $\bar{A}$ and $\bar{B}$ be as above. Assume further that A_1 is continuously embedded in A_0 or, alternatively, B_1 is continuously embedded in B_0, and that $T : \bar{A} \to \bar{B}$ with*

$$T : A_1 \to B_1 \quad \textit{compactly}.$$

Then

$$T : G(\bar{A}) \to G(\bar{B}) \quad \textit{is compact}.$$

In order to show some concrete cases, let l^q $(1 \leqq q \leqq \infty)$ be the usual space of doubly infinite scalar sequences and, for $\alpha \in \mathbf{R}$, define $l^q(2^{\alpha j})$ by

$$l^q(2^{\alpha j}) = \{\{\xi_j\} \mid \{2^{\alpha j}\xi_j\} \in l^q\}.$$

EXAMPLE 2.4. If $\bar{X} = (l^1, l^1(2^{-j}))$ and $X = l^q(2^{-\theta j})$ [where $0 < \theta < 1$, $1 \leqq q \leqq \infty$] then, for any Banach couple $\bar{A}$,

$$G[l^1, l^1(2^{-j}); l^q(2^{-\theta j})](\bar{A}) = \bar{A}_{\theta,q}$$

(one of the spaces of the real method, realized here as a J-space; see [4], [14], [19]). Thus we have an interpolation functor of exponent θ and the approximation hypotheses are fulfilled: We define

$$P_n \xi = (\ldots, 0, \xi_{-n}, \ldots, \xi_{-1}, \xi_0, \xi_1, \ldots, \xi_n, 0, \ldots)$$

for any doubly infinite sequence

$$\xi = (\ldots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \ldots).$$

Similarly, we set

$$Q_n^+ \xi = (\ldots, 0, 0, \xi_{n+1}, \xi_{n+2}, \ldots)$$

and

$$Q_n^- \xi = (\ldots, \xi_{-n-2}, \xi_{-n-1}, 0, 0, \ldots).$$

Then (I) and (II) are clear. To prove (III) let $\xi \in X_0 = l^1$. We have

$$\| Q_n^+ \xi \|_{X_1} = \sum_{j=n+1}^{\infty} 2^{-j} |\xi_j| \leqq 2^{-n} \sum_{j=n+1}^{\infty} |\xi_j|$$

$$\leqq 2^{-n} \sum_{j=-\infty}^{\infty} |\xi_j| = 2^{-n} \| \xi \|_{X_0}.$$

In a similar way we check that

$$\| Q_n^- \xi \|_{X_0} \leqq 2^{-n} \| \xi \|_{X_1}$$

for $\xi \in X_1$.

When we write Theorem 2.2 for this example, we obtain the Banach case $(1 \leqq q \leqq \infty)$ of [6], Thm. 3.1, while Theorem 2.3 when $A_1 \hookrightarrow A_0$ gives [7], Thm. 2.1. The case $B_1 \hookrightarrow B_0$ in Theorem 2.3 produces a new result.

EXAMPLE 2.5. In the same way as above, one sees that the approximation condition is fulfilled with l^1 replaced by the space c_0 of sequences tending to 0 at infinity. The resulting functor (see [14])

$$G[c_0, c_0(2^{-j}); c_0(2^{-\theta j})](-) \quad \text{equals} \quad \langle - \rangle_\theta$$

(also called the " $\pm$ " method; see [22] and [12]). When writing down Theorems

2.2 and 2.3 for the " $\pm$ " method, we obtain the first known results for this functor on the behaviour of compactness without an approximation condition.

REMARK 2.6. The complex method [5] can be also described as a minimal functor

$$G[\mathrm{FL}, \mathrm{FL}(2^{-j}); \mathrm{FL}(2^{-\theta j})](\bar{A}) = \bar{A}_\theta.$$

However, as we already indicated in the Introduction, the space FL, consisting of Fourier coefficients of Lebesgue integrable functions, does not behave as the former ones. Following [15] one might think that smoothed out versions of the previous operators P_n, Q_n^+, Q_n^- might work. Well, everything is fine concerning P_n but we have not been able to construct any workable analogue of Q_n^+, Q_n^-. (A similar difficulty has been encountered previously (see [9]).)

3. The case of the Aronszajn–Gagliardo maximal functor

Next we turn to the dual results formally gotten from the theorems in Section 2 by reversing all arrows.

Let us start by recalling how the Aronszajn–Gagliardo maximal ("coorbit") functor [1] is defined, the construction dual to the one in Section 2.

Set now

$$V = V(\bar{B}) = \{R \mid R : \bar{B} \to \bar{X},\ \| R \|_{\bar{B},\bar{X}} \leqq 1\}$$

and let $l^\infty[Y] = l^\infty(V(\bar{B}), Y)$ have a similar meaning as $l^1[Y]$ in Section 2: it is the Banach space consisting of all bounded Y-valued families $\{y_R\}$ with V as index set, normed by

$$\| y \|_{l^\infty[Y]} = \sup_{R \in V} \| y_R \|_Y.$$

We further set

$$ib = \{Rb\}_{R \in V} \qquad \text{for } b \in B_0 + B_1,$$

thus ib belongs to $l^\infty[X_0 + X_1]$, and we define

$$H(\bar{B}) = H[\bar{X}; X](\bar{B}) = \{b \mid b \in B_0 + B_1,\ ib \in l^\infty[X]\};$$

it is a Banach space in the natural induced norm and if $T : \bar{A} \to \bar{B}$ then $T : H(\bar{A}) \to H(\bar{B})$. Hence H is an interpolation functor. As in Section 2, we assume for convenience that it is of exponent θ (where $0 < \theta < 1$).

About the given entities $\bar{X} = (X_0, X_1)$ and X we make the same assumptions as in the preceding section, viz. the approximation hypotheses (I) to (III).

Again, it is *vital* for our approach that the operators P_n, Q_n^+, Q_n^- mentioned there extend in a natural way to $l^\infty[X_0 + X_1]$ preserving properties (I) to (III).

Now we establish an auxiliary result.

LEMMA 3.1. *Let, similarly as in Lemma 2.1, $l^\infty(\cdot)$ be l^∞ with respect to an arbitrary fixed index set $\mathcal{N}$ (i.e., if B is any Banach space then $l^\infty(B)$ consists of all B-valued families $b = \{b_\nu\}$ such that $\sup_{\nu\in\mathcal{N}} \| b_\nu \|_B < \infty$). Then for any Banach couple $\bar{B} = (B_0, B_1)$*

$$H(l^\infty(B_0), l^\infty(B_1)) \hookrightarrow l^\infty(H(\bar{B}))$$

holds.

PROOF. Let $b = \{b_\nu\}$ be in $H(l^\infty(B_0), l^\infty(B_1))$ of norm 1. If R is any element of $V = V(\bar{B})$, that is a linear map $R : \bar{B} \to \bar{X}$ of norm $\| R \|_{\bar{B},\bar{X}} \leqq 1$, and ν_0 is any fixed index in $\mathcal{N}$, we define a linear map

$$R_{\nu_0} : (l^\infty(B_0), l^\infty(B_1)) \to \bar{X}$$

of norm $\leqq 1$, by stipulating that

$$R_{\nu_0}\{y_\nu\} = Ry_{\nu_0}.$$

By the definition of $H(l^\infty(B_0), l^\infty(B_1))$, we have that

$$R_{\nu_0}\{b_\nu\} = Rb_{\nu_0} \in X \qquad \text{with } \| Rb_{\nu_0} \|_X \leqq 1.$$

Now, as R is arbitrary in this reasoning, it follows that

$$b_{\nu_0} \in H(\bar{B}) \quad \text{and} \quad \| b_{\nu_0} \|_{H(\bar{B})} \leqq 1.$$

Finally, since ν_0 is also arbitrary, we obtain that

$$b = \{b_\nu\} \in l^\infty(H(\bar{B})) \qquad \text{with } \| b \|_{l^\infty(H(\bar{B}))} \leqq 1. \qquad \square$$

Next we discuss compactness results for the coorbit functor.

THEOREM 3.2. *Let the hypotheses* (I) *to* (III) *be fulfilled for $\bar{X}$ and assume also that $H[\bar{X}; X](-)$ is an interpolation functor of exponent θ $(0 < \theta < 1)$. Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be any two Banach couples, and let $T : \bar{A} \to \bar{B}$ be such that*

$$T : A_0 \to B_0 \quad \textit{and} \quad T : A_1 \to B_1$$

are both compact. Then

$$T : H(\bar{A}^\circ) \to H(\bar{B}^\circ) \quad \textit{is compact}.$$

PROOF. First note that it suffices to show that $\tilde{\tilde{T}} = iT$ is compact. The relevant diagram reads now

$$A_0^\circ \xrightarrow{T} B_0^\circ \xrightarrow{i} l^\infty[X_0]$$

$$A_1^\circ \xrightarrow{T} B_1^\circ \xrightarrow{i} l^\infty[X_1]$$

$$\overline{\overline{\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$H(\bar{A}^\circ) \xrightarrow{T} H(\bar{B}^\circ) \xrightarrow{i} l^\infty[X]$$

According to Theorem 1.8, we have that

$$\tilde{\tilde{T}} : H(\bar{A}^\circ) \to H(l^\infty[X_0], l^\infty[X_1])$$

is compact, and by Lemma 3.1, we know that

$$H(l^\infty[X_0], l^\infty[X_1]) \hookrightarrow l^\infty[H(X_0, X_1)] \hookrightarrow l^\infty[X].$$

Consequently

$$\tilde{\tilde{T}} : H(\bar{A}^\circ) \to l^\infty[X] \quad \text{compactly.}$$

□

In the case of ordered couples we obtain a onesided result as well.

THEOREM 3.3. *Let $\bar{X}$, H, $\bar{A}$ and $\bar{B}$ as above. Suppose further that A_0 is continuously embedded in A_1 or alternatively B_0 is continuously embedded in B_1, and that $T : \bar{A} \to \bar{B}$ with*

$$T : A_1 \to B_1 \quad \textit{compactly.}$$

Then

$$T : H(\bar{A}^\circ) \to H(\bar{B}^\circ) \quad \textit{is compact.}$$

PROOF. Is the same as the previous one but replacing Theorem 1.8 by Theorem 1.9.

Let us now see a concrete case.

EXAMPLE 3.4. Take $\bar{X} = (l^\infty, l^\infty(2^{-j}))$ and $X = l^q(2^{-\theta j})$ (where $0 < \theta < 1$, $1 \leqq q \leqq \infty$). Then, for any Banach couple $\bar{A}$, we have

$$H[l^\infty, l^\infty(2^{-j}); l^q(2^{-\theta j})](\bar{A}) = H[l^\infty, l^\infty(2^{-j}); l^q(2^{-\theta j})](\bar{A}^0) = \bar{A}_{\theta,q}$$

again the real method, but this time in the form of a K-space (see [14], [19]). And the approximation condition is fulfilled by $\bar{X}$ as one can see proceeding analogously to Example 2.4.

For this concrete example, Theorem 3.3 when $B_0 \hookrightarrow B_1$ is [6], Thm. 3.2. The case $A_0 \hookrightarrow A_1$ gives a new result.

Let us write down the new information contained in Examples 2.4 and 3.4.

COROLLARY 3.5. *Assume that* $\bar{A} = (A_0, A_1)$ *and* $\bar{B} = (B_0, B_1)$ *are Banach couples with any of them ordered, and let T be a linear operator such that*

$$T: A_0 \to B_0 \quad \textit{is compact}$$

and

$$T: A_1 \to B_1 \quad \textit{is bounded.}$$

Then if $0 < \theta < 1$ *and* $1 \leqq q \leqq \infty$,

$$T: \bar{A}_{\theta,q} \to \bar{B}_{\theta,q} \quad \textit{is compact.}$$

We remark that this corollary allows us to incorporate the case $q_0 = \infty$ to Krasnosel'skiĭ's result mentioned in the Introduction. We only need to require that any of the two measure spaces, where the operator is defined, has finite measure. In [2], Thm. IV.2.9, one can find another proof for the case $q_0 = \infty$ when the second measure space is finite.

4. Concluding observations

In this last section we indicate briefly some directions in which the present investigation might be continued.

(a) First of all that the theorems of Sections 2 and 3 probably can be generalized to a multidimensional case. In particular, this would yield concrete compact interpolation theorems for the functors of Sparr [24] and Fernandez [10]. At least for the former a representation as Aronszajn–Gagliardo functors is already available [8].

(b) Next, we have already said that, contrary to what we intitially thought, our approach seems to fail in the case of complex interpolation. Perhaps this is an indication of the fact that there are no compact interpolation theorems without approximation hypotheses in this case. But it is probably very hard to find a counterexample.

REMARK 4.1. A way out of the difficulty (the nonexistence of a decomposition $I = P_n + Q_n^+ + Q_n^-$; see Section 2) would be to replace FL by the space FL_+ (singly, not doubly infinite sequences). This means that one essentially restricts oneself to ordered pairs $\bar{A} = (A_0, A_1): A_0 \supset A_1$. The difficulty is however that then we do not know if FL and FL_+ yield the same spaces (cf. [9]).

(c) Finally, of course, what still remains open is the question if there exist onesided (that is, Krasnosel'skiĭ type) compact interpolation theorems without any auxiliary condition (i.e., approximation hypothesis or ordered couples). When we finished this paper (October 88) we thought that perhaps this would not be the case. We even tried to find a counterexample for the real method, looking at weighted L^∞-couples (this, because one knows [20], [21] that a general couple $\bar{B}$ may be viewed as a subcouple of such a couple). But now (May 89) we have been kindly informed by M. Cwikel that a Krasnosel's-kiĭ-type theorem holds for the real method.

REFERENCES

1. N. Aronszajn and E. Gagliardo, *Interpolation spaces and interpolation methods*, Ann. Mat. Pura Appl. **68** (1965), 51–118.
2. C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, London, 1988.
3. J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin–Heidelberg–New York, 1976.
4. Yu. A. Brudnyĭ and N. Ja. Kruglyak, *Real Interpolation Functors*, North-Holland, Amsterdam, to appear.
5. A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190.
6. F. Cobos, D. E. Edmunds and A. J. B. Potter, *Real interpolation and compact linear operators*, J. Funct. Anal., to appear.
7. F. Cobos and D. L. Fernandez, *On interpolation of compact operators*, Ark. Mat., to appear.
8. F. Cobos and J. Peetre, *A multidimensional Wolff's theorem*, Studia Math., to appear.
9. M. Cwikel, M. Milman and J. Peetre, *Complex extrapolation*, in preparation.
10. D. L. Fernandez, *Interpolation of 2^d Banach spaces and the Calderón spaces $X(E)$*, Proc. London Math. Soc. **56** (1988), 143–162.
11. E. Gagliardo, *Una structtura unitaria in diverse famiglie di spazi funzionali*, Ricerche Mat. **10** (1961), 244–281.
12. J. Gustavsson and J. Peetre, *Interpolation of Orlicz spaces*, Studia Math. **60** (1977), 33–59.
13. K. Hayakawa, *Interpolation by the real method preserves compactness of operators*, J. Math. Soc. Japan **21** (1969), 189–199.
14. S. Janson, *Minimal and maximal methods of interpolation*. J. Funct. Anal. **44** (1981), 50–73.
15. S. Janson, P. Nilson, J. Peetre and M. Zafran, *Notes on Wolff's note on interpolation spaces*, Proc. London Math. Soc. **48** (1984), 283–299.
16. M. A. Krasnosel'skiĭ, *On a theorem of M. Riesz*, Soviet Math. Dokl. **1** (1960), 229–231.
17. S. G. Kreĭn and Ju. I. Petunin, *Scales of Banach spaces*, Russian Math. Surveys **21** (1966), 85–159.
18. J. L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Etudes Sci. Publ. Math. **19** (1964), 5–68.
19. V. I. Ovchinnikov, *The Method of Orbits in Interpolation Theory*, Math. Reports **1** (1984), 349–515.

20. J. Peetre, *Banach couples I. Elementary theory*, Technical report, Lund, 1971.

21. J. Peetre, *Interpolation functors and Banach couples*, in *Actes Congrès Internat. Math.* 1970, Vol. 2, Gauthier-Villars, Paris, 1971, pp. 373–378.

22. J. Peetre, *Sur l'utilisation des suites inconditionellement sommables dans la théorie des espaces d'interpolation*, Rend. Sem. Mat. Univ. Padova **46** (1971), 173–190.

23. A. Persson, *Compact linear mappings between interpolation spaces*, Ark. Mat. **5** (1964), 215–219.

24. G. Sparr, *Interpolation of several Banach spaces*, Ann. Mat. Pura Appl. **99** (1974), 247–316.

25. T. Wolff, *A note on interpolation spaces*, in *Proceedings Conference on Harmonic Analysis, Minneapolis 1981*, Lecture Notes in Mathematics **908**, Springer, Berlin, 1982, pp. 199–204.